Math 434 Assignment 4

Due May 10

Assignments will be collected in class.

1. In class we proved Ramsey's theorem for pairs and two colours. Use this to prove Ramsey's theorem for pairs and n colours:

   Let k be a natural number, and let $c\colon [\omega]^2 \to k$ be a function. Then there is an infinite subset $H \subseteq \omega$ such that c is constant on subsets of H.

2. A filter $\mathcal{F}$ on ω is said to be countably generated if there are sets $\{A_i\}_{i\in\omega}$ such that

   $$\mathcal{F} = \{X \subseteq \omega : (\exists i \in \omega)\ A_i \subseteq X\}.$$

   Prove that no non-principal ultrafilter can be countably generated.

3. Let $\mathcal{F}$ be the principal ultrafilter on I generated by $j \in I$. Show that $\prod_I \mathcal{M}_i/\mathcal{F} \cong M_j$.

4. Let T be a complete theory. A *complete n-type (of T)* $p(x_1, \ldots, x_n)$ is a set of formulas in n free variables $x_1, \ldots, x_n$ such that:

   - for every formula $\psi(x_1, \ldots, x_n) \in p$, $T \vDash \exists x_1, \ldots, x_n \psi(x_1, \ldots, x_n)$;
   - for every formula $\psi(x_1, \ldots, x_n)$, either $\psi \in p$ or $\neg\psi \in p$.

   We say that a model $\mathcal{M}$ realizes a complete n-type p if there are $a_1, \ldots, a_n \in \mathcal{M}$ such that for every $\psi \in p$, $\mathcal{M} \vDash \psi(a_1, \ldots, a_n)$.

   Show that there is a model of T that realizes every complete n-type of T.

5. Prove that if α is a limit ordinal, then the axiom of union holds in V_α:

   $$V_\alpha \vDash (\forall x)(\exists y)(\forall z)[z \in y \longleftrightarrow (\exists w \in x)(z \in w)].$$

6. A strongly inaccessible cardinal is an uncountable regular limit cardinal κ such that for all $\lambda < \kappa$, $2^\lambda < \kappa$. Prove that if κ is strongly inaccessible, then for all $\alpha < \kappa$, $|V_\alpha| < \kappa$.

7. Except for the axiom of infinity, all the axioms of ZFC hold in V_ω. Prove that the axiom of infinity does not hold, i.e., $V_\omega \vDash \neg(\exists x)[\varnothing \in x \wedge (\forall y \in x)(\{y\} \cup y \in x)]$. (This shows that the axiom of infinity is not implied by the remaining axioms of ZFC.)